arXiv:1301.2960v1 [math.MG] 14 Jan 2013

A universality theorem for projectively unique polytopes and a conjecture of Shephard

Karim A. Adiprasito *
Institut für Mathematik, FU Berlin
Arnimallee 2
14195 Berlin, Germany
adiprasito@math.fu-berlin.de

Arnau Padrol‡
Institut für Mathematik, FU Berlin
Arnimallee 2
14195 Berlin, Germany
arnau.padrol@upc.edu

January 15, 2013

Abstract

We prove that every rational polytope is the face of a projectively unique polytope. As a corollary, we provide a projectively unique polytope that is not the subpolytope of any stacked polytope. This disproves a classical conjecture in polytope theory, first formulated by Shephard in the seventies.

Using a technique developed by Adiprasito and Ziegler [AZ12], we prove the following universality theorem for projectively unique polytopes.

Theorem 1. *If P is any polytope with rational vertex coordinates, then there exists a polytope P' that is projectively unique and contains a face projectively equivalent to P.*

Here, a polytope P in $\mathbb{R}^d$ is *projectively unique* if any polytope P' in $\mathbb{R}^d$ combinatorially equivalent to P is *projectively equivalent* to P. In other words, for every P' combinatorially equivalent to P, there exists a projective transformation T of $\mathbb{R}^d$ that realizes the given combinatorial isomorphism from P to P'.

We apply this result to a classical conjecture of Shephard. He considered the question whether every polytope is a *subpolytope* of a stacked polytope, i.e. whether it can be obtained as the convex hull of a subset of the vertices of some stacked polytope. While he proved this wrong in [She74], he conjectured it to be true in a combinatorial sense:

Conjecture 2 (Shephard [She74], Kalai [Kal04, p. 468], [Kal12])**.** *Every combinatorial type of polytope can be realized by a subpolytope of a stacked polytope.*

The conjecture is true for 3-dimensional polytopes, as seen by Kömhoff in [Kö80], but remained open in all higher dimensions. We use Theorem 1 to disprove the conjecture by finding a high-dimensional counterexample.

Theorem 3. *There exists a projectively unique polytope that is not a subpolytope of any stacked polytope.*

Since any admissible projective transformation of a stacked polytope is a stacked polytope, no realization of the polytope announced in Theorem 3 is a subpolytope of a stacked polytope. Hence, the preceding theorem clearly provides a counterexample to Conjecture 2.

Corollary 4. *There exists a combinatorial type of polytope that can not be realized as a subpolytope of any stacked polytope.*

*Supported by DFG within the research training group "Methods for Discrete Structures" (GRK1408) and by the Romanian NASR, CNCS – UEFISCDI, project PN-II-ID-PCE-2011-3-0533.
‡Supported by the DFG Collaborative Research Center SFB/TR 109 "Discretization in Geometry and Dynamics"

Proof of Theorem 1

Point configurations, PP configurations and weak projective triples

We recall the basic facts about projectively unique point configurations and polytope-point configurations, compare also [AZ12, Sec. 5.1 & 5.2], [Grü03, Sec. 4.8 Ex. 30], [RG96, Part I].

Definition 5 (PP configurations, Lawrence equivalence, projective uniqueness)**.** A *point configuration* is a finite collection R of points in $\mathbb{R}^d$. If H is an oriented hyperplane in $\mathbb{R}^d$, then we use H_+ resp. H_- to denote the open halfspaces bounded by H. If P is a polytope in $\mathbb{R}^d$ such that $P \cap R = \emptyset$ then the pair (P, R) is a *polytope–point configuration*, or *PP configuration.* A hyperplane H is *external to* P if $H \cap P$ is a face of P.

Two PP configurations (P, R), (P', R') in $\mathbb{R}^d$ are *Lawrence equivalent* if there is a bijection φ between the vertex sets of P and P' and the sets of R and R', such that, if H is any hyperplane such that the closure of H_- contains P, there exists an oriented hyperplane H' such that the closure of H'_- contains P' and

$$\varphi(\mathrm{F}_0(P) \cap H_-) = \mathrm{F}_0(P') \cap H'_-, \qquad \varphi(R \cap H_+) = R' \cap H'_+, \qquad \varphi(R \cap H_-) = R' \cap H'_-.$$

A PP configuration (P, R) in $\mathbb{R}^d$ is *projectively unique* if for any PP configuration (P', R') in $\mathbb{R}^d$ Lawrence equivalent to it, and every bijection ϕ that induces the Lawrence equivalence, there is a projective transformation T that realizes ϕ. A point configuration R is *projectively unique* if the PP configuration $(\emptyset, R)$ is projectively unique, and it is not hard to verify that a polytope P is projectively unique if the PP configuration $(P, \emptyset)$ is projectively unique.

Proposition 6 (Lawrence extensions, cf. [RG96, Lem. 3.3.3 and 3.3.5], [Zie08, Thm. 5])**.** *Let (P, R) be a projectively unique PP configuration in $\mathbb{R}^d$. Then there exists a $(\dim P + f_0(R))$-dimensional polytope on $f_0(P) + 2f_0(R)$ vertices that is projectively unique and that contains P as a face.*

Definition 7 (Framed polytopes)**.** Let P denote a polytope in $\mathbb{R}^d$, and let Q be any subset of its vertex set $\mathrm{F}_0(P)$, that is, $Q = \{q_1, q_2, q_3, \dots\} \subseteq \mathrm{F}_0(P)$. Let P' be any polytope in $\mathbb{R}^d$ combinatorially equivalent of P. Let φ denote the labeled isomorphism from the faces of P to the faces of P'. We say that the polytope P *is framed by* the set of vertices Q if $P = P'$ for all choices of P' and φ that satisfy $\varphi(q) = q$ for all $q \in Q$.

Examples 8. We record some instances of framed polytopes.

(i) If P is any polytope, then $\mathrm{F}_0(P)$ frames P.

(ii) If P is a projectively unique polytope, and $Q \subseteq \mathrm{F}_0(P)$ is a projective basis for its span, then Q frames P.

(iii) A 3-cube W is framed by any 7 of its vertices. Similarly, any d-cube, $d \geq 3$, is framed by $2^d - 1$ of its vertices.

Definition 9 (Weak projective triple in $\mathbb{R}^d$)**.** A triple (P, Q, R) of a polytope P in $\mathbb{R}^d$, a subset Q of $\mathrm{F}_0(P)$ and a point configuration R in $\mathbb{R}^d$ is a *weak projective triple* in $\mathbb{R}^d$ if and only if

(1) $(\emptyset, Q \cup R)$ is a projectively unique point configuration,

(2) Q frames the polytope P, and

(3) some subset of R spans a hyperplane H, called the *wedge hyperplane*, which does not intersect P.

Definition 10 (Subdirect Cone)**.** Let (P, Q, R) be a weak projective triple in $\mathbb{R}^d$, seen as canonical subspace of $\mathbb{R}^{d+1}$. Let H denote the wedge hyperplane in $\mathbb{R}^d$ spanned by vertices of R with $H \cap P = \emptyset$. Let v denote any point not in $\mathbb{R}^d$, and let $\widehat{H}$ denote any hyperplane in $\mathbb{R}^{d+1}$ such that $\widehat{H} \cap R^d = H$ and $\widehat{H}$ separates v from P. Consider, for every vertex p of P, the point $p^v = \mathrm{conv}\{v, p\} \cap \widehat{H}$. Denote by P^v the pyramid

$$P^v := \mathrm{conv}\Big(v \cup \bigcup_{p \in \mathrm{F}_0(P)} p^v\Big).$$

The PP configuration $(P^v, Q \cup R)$ in $\mathbb{R}^{d+1}$ is a *subdirect cone* of (P, Q, R).

Lemma 11 ([AZ12, Lemma 5.8.])**.** *For any weak projective triple (P, Q, R) the subdirect cone $(P^v, Q \cup R)$ is a projectively unique PP configuration.*

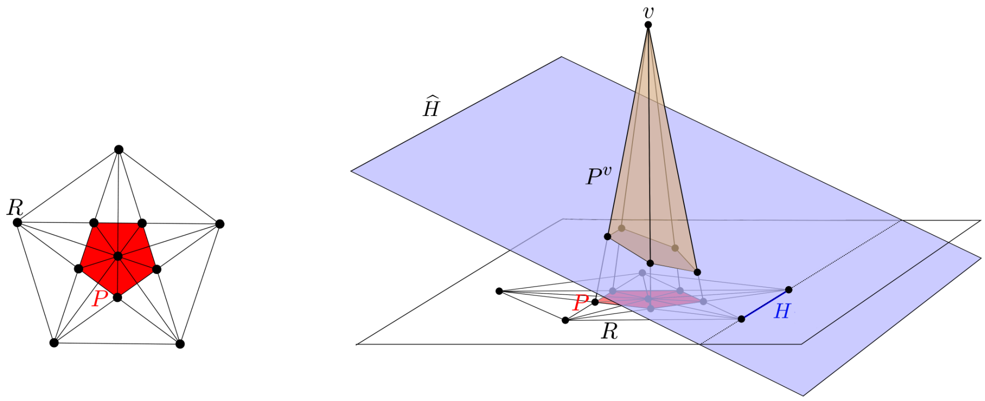


Figure 1: The subdirect cone of the weak projective triple (P, Q, R) in the special case where Q coincides with the vertexset of P

Corollary 12. *If (P, Q, R) is a weak projective triple, there exists a projectively unique polytope of dimension $\dim P + \#Q + \#R + 1$ that contains a face projectively equivalent to P.*

Proof. By Lemma 11, the subdirect cone $(P^v, Q \cup R)$ is a projectively unique PP configuration, and by construction, the polytope P^v (of dimension $\dim P + 1$) has a facet projectively equivalent to P. Consequently, by Proposition 6, there exists a projectively unique polytope of the desired dimension with a face projectively equivalent to P. □

Conclusion of proof

The proof of Theorem 1 relies on the following instance of a projectively unique point configuration.

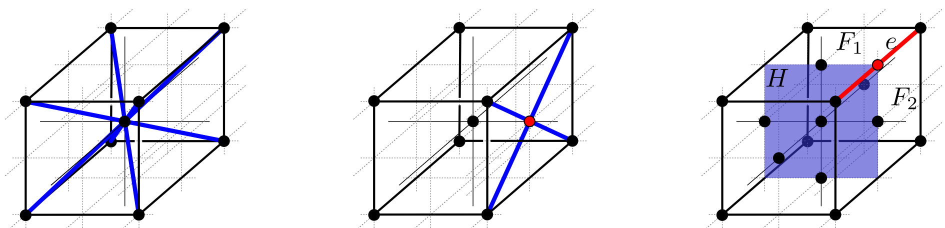
Proposition 13. *Let m be any nonnegative integer, and $d \geq 3$. The set Q_m^d defined as*

$$Q_m^d := \{v \in \mathbb{Z}^d \subset \mathbb{R}^d : ||v||_\infty \leq 2^m\}$$

is a projectively unique point configuration.

Proof. The proof is by induction on d and m. We start proving that Q_0^3 is projectively unique. This implies that Q_0^d is projectively unique for any $d \geq 3$, and finally that Q_m^d is projectively unique for any $d \geq 3$ and $m \geq 0$.

Q_0^3 is projectively unique: To see that Q_0^3 is projectively unique, we start with the folklore observation that the vertices $(\pm1, \pm1, \pm1)$, together with the origin $(0, 0, 0)$, form a projectively unique configuration W (cf. Figure 2a). Clearly, this point configuration is a subset of Q_0^3. Furthermore, we claim that all remaining vertices of Q_0^3 can be determined from W by affine dependencies only, thereby proving that Q_0^3 is projectively unique since W is projectively unique.



(a) W: The vertices of $[-1, 1]^3$ and the origin.

(b) Adding the center-points of the facets.

(c) Adding the mid-points of the edges.

Figure 2: Showing that Q_0^3 is projectively unique.

To see this, notice that the point $(1, 0, 0)$ of $Q_0^3 \setminus W$ is determined as the intersection of the lines $\operatorname{aff}\{(+1, +1, +1), (+1, -1, -1)\}$ and $\operatorname{aff}\{(+1, +1, -1), (+1, -1, +1)\}$, which are spanned by vertices of W.

Similarly, all vertices that arise as coordinate permutations and/or sign changes from $(+1, 0, 0)$ are determined this way. Geometrically, these are the centerpoints of the facets of the cube $[-1, 1]^3 = \operatorname{conv} W = \operatorname{conv} Q_0^3$ (cf. Figure 2b).

The remaining vertices of Q_0^3 coincide with the midpoints of the edges of said cube. To determine them, let e be any edge of Q_0^3 and let F_1 and F_2 be the facets of Q_0^3 incident to that edge. Finally, let H be the hyperplane spanned by the centerpoint of Q_0^3 and the centerpoints of F_1 and F_2. The midpoint of e is the unique point of intersection of e and H (cf. Figure 2c).

Q_0^d is projectively unique: For $d \geq 4$, consider the projective basis B of $\mathbb{R}^d$ consisting of the vertex $v_0 := (+1, +1, \dots, +1)$ of $[-1, 1]^d$, together with the neighboring vertices $v_1 := (-1, +1, \dots, +1)$, $v_2 := (+1, -1, \dots, +1)$, $\dots$, $v_d := (+1, +1, \dots, -1)$ and the origin $o := (0, \dots, 0)$ (cf. Figure 3a). We will see that once the coordinates of the elements in B are fixed, then the coordinates of all the remaining points of Q_0^d can be determined uniquely.

Consider the set of points of Q_0^d lying in a common facet of $[-1, 1]^d$ that is incident to v_0; for example, $R_1 := Q_0^d \cap \operatorname{aff}\{v_0, v_2, \dots, v_d\}$ (cf. Figure 3b). Observe that R_1 is just an affine embedding of Q_0^{d-1} into $\mathbb{R}^d$. As such, R_1 is projectively unique, and thus it is determined uniquely if a projective basis for its span is fixed.

Clearly, the points $v_0, v_2, \dots, v_d$ of B form an affine basis for the span of R_1. Furthermore, the coordinates of the point $w = (+1, -1, \dots, -1)$, are fixed by B. Indeed, w is the the point of intersection of the line $\operatorname{aff}\{o, v_1\}$ with the hyperplane $\operatorname{aff}\{R_1\}$ (cf. Figure 3c). To sum up, we have that

- the points $v_0, v_2, \dots, v_d, w$ are determined from the points of B by affine dependencies only,
- the points $v_0, v_2, \dots, v_d, w$ are elements of R_1, and
- the points $v_0, v_2, \dots, v_d, w$ form a projective basis for the span of R_1.

Consequently, $R_1 \cup B$ is a projectively unique point configuration, since B is projectively unique. We can repeat this argumentation for all point configurations

$$R_i := Q_0^d \cap \operatorname{aff}\left(\{v_0, v_1, v_2, \dots, v_d\} \setminus \{v_i\}\right), \ i \in \{1, \dots, d\}.$$

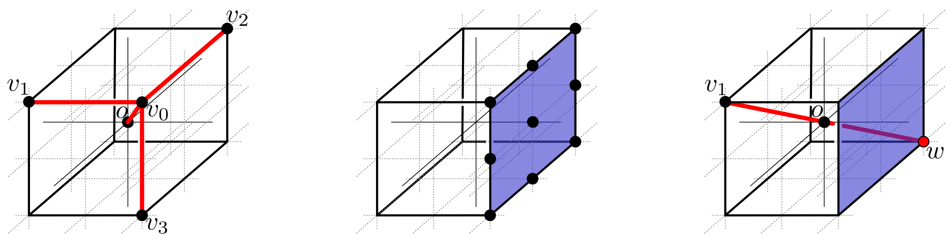
In particular, the configuration

$$\widetilde{Q}_0^d = B \cup \bigcup_{i \in \{1, \dots, d\}} R_i$$

is projectively unique. Moreover, since the last vertex of a cube of dimension $d \geq 3$ is determined by the remaining ones by affine dependencies (cf. [AZ12, Lemma 3.4], compare also Example 8(iii)), the configuration $\widetilde{Q}_0^d \cup \{-v_0\}$ is projectively unique as well. By symmetry, the point configuration

$$-\widetilde{Q}_0^d \cup \{v_0\} = -B \cup \bigcup_{i \in \{1, \dots, d\}} -R_i \cup \{v_0\}$$

is also projectively unique.



(a) B **(b)** R_1 **(c)** $\operatorname{aff}\{o, v_1\} \cap \operatorname{aff}\{R_1\}$

Figure 3: Schema for showing that Q_0^d is projectively unique. (The picture displays Q_0^3 for the sake of clarity, but the proof starts with $d \geq 4$)

Clearly, $\widetilde{Q}_0^d \cup \{-v_0\}$ and $-\widetilde{Q}_0^d \cup \{v_0\}$ intersect along a projective basis: for instance, the set B lies in both $-\widetilde{Q}_0^d \cup \{v_0\}$ and $\widetilde{Q}_0^d \cup \{-v_0\}$ and forms a projective basis as desired. Thus, the point configuration $\widetilde{Q}_0^d \cup \{-v_0\} \cup -\widetilde{Q}_0^d \cup \{v_0\}$ is projectively unique.

Q_m^d **is projectively unique:** For higher values of m, the proof is recursive. Observe first that the points of Q_m^d correspond to the half-integer points in $\operatorname{conv} Q_{m-1}^d$ (cf. Figure 4), and that Q_{m-1}^d is projectively unique by induction hypothesis. Moreover, once the realization of Q_{m-1}^d is fixed, the coordinates of each of the remaining points of Q_m^d can be determined. Indeed, each $1 \times 1 \times 1$ cell of Q_m^d is a shrunken copy of Q_0^d. Since the coordinates of the vertices of this cell are already fixed and contain a projective basis of $\mathbb{R}^d$, the coordinates of all half-integer points in the cell can be determined because Q_0^d is projectively unique.

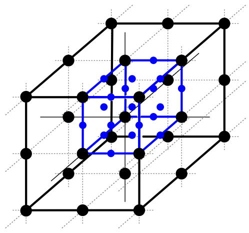


Figure 4: Showing that Q_m^3 is projectively unique.

$\square$

Proof of Theorem 1. Let P denote a polytope in $\mathbb{R}^d$ whose vertices have rational coordinates. We assume that $d \geq 3$, since if $d \leq 2$ we can embed P as a face of a 3-dimensional pyramid. Dilating with a suitable integer factor, we may assume that all vertices of P lie in $\mathbb{Z}^d$. Furthermore, we can choose some $m \geq 0$ large enough so that the vertices of P can be confined to $\operatorname{vert}(P) \subset Q_m^d \subset \mathbb{Z}^d$, and such that some subset of points of Q_m^d spans a hyperplane that does not intersect P (this can be, for example, a facet hyperplane of $\operatorname{conv} Q_m^d$). Consider the triple (P, Q, R), where Q denotes the vertices of P, and $R := Q_m^d \setminus Q$. Then (P, Q, R) is a weak projective triple, since
(1) $Q \cup R = Q_m^d$ is a projectively unique point configuration by Proposition 13,
(2) Q obviously frames P, and
(3) R, by construction, has a subset that spans a hyperplane that does not intersect P.
Thus, by Corollary 12, there exists a projectively unique polytope (of dimension $d + (2^{m+1}+1)^d + 1$) that contains a face projectively equivalent to P. $\square$

Subpolytopes of stacked polytopes

Our methods make it more convenient to work with the dual formulation of Conjecture 2. We recall some notions:

Definition 14 (Dual-to-stacked polytopes)**.** A polytope is *dual-to-stacked* if it can be obtained from a simplex by a sequence of vertex truncations. A *vertex truncation* is the intersection of a polytope with a halfspace that cuts off exactly one of its vertices (cf. Figure 5).

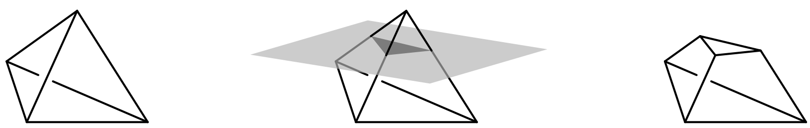


Figure 5: Truncation of a vertex.

It now only remains to introduce the dual notion to subpolytopes, and apply it to stacked polytopes.

Definition 15 (Substacked polytopes)**.** We call a polytope *substacked* if it can be obtained from a dual-to-stacked polytope by removing some facet-defining inequalities.

Clearly, a polytope (that contains the origin in the interior) is substacked if and only if its polar dual is the subpolytope of a stacked polytope. The proof of Theorem 3 relies on the following simple fact:

Lemma 16. *All faces of substacked polytopes are substacked.* $\square$

Thus, to finish the proof of Theorem 3 it is enough to provide a projectively unique polytope one of whose faces is not substacked. The combination of the following proposition, together with Theorem 1 and Lemma 16, clearly implies Theorem 3.

Proposition 17. *There exists a polytope in $\mathbb{R}^3$ with rational vertex coordinates that is not substacked.*

Proof. An ε-net N_ε in a metric space X is a set of points in X with the property that no point of X is farther than ε from an element of N_ε. Recall the following classical results:

- *Shephard,* [She74]: If $\varepsilon > 0$ is sufficiently small, then for any ε-net N_ε in the unit sphere $S^2 \subset \mathbb{R}^3$ (with respect to the euclidean metric on $\mathbb{R}^3$), the polytope $\operatorname{conv} N_\varepsilon$ is not the subpolytope of any stacked polytope.
- *Classical fact, cf.* [HW54, Sec. XIII], [Duk03]: Points with rational coordinates are dense in S^2. In particular, for any $\varepsilon > 0$ there is an ε-net N_ε in S^2 whose points have rational coordinates.

Finally, the polar dual of a polytope with rational vertex coordinates has rational vertex coordinates. Thus, if N is a sufficiently dense set of rational points in S^2, then $(\operatorname{conv} N)^*$, the polar dual to $\operatorname{conv} N$, is not substacked and has vertices in $\mathbb{Q}^3$. $\square$

References

[AZ12] K. Adiprasito and G. M. Ziegler, *A polytope that is not a subpolytope of any stacked polytope*, preprint, available at arxiv.org/abs/1212.5812.

[Duk03] W. Duke, *Rational points on the sphere*, Ramanujan J. **7** (2003), no. 1-3, 235–239, Rankin memorial issues.

[Grü03] B. Grünbaum, *Convex Polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer, New York, 2003.

[HW54] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, at the Clarendon Press, 1954, 3rd ed.

[Kal04] G. Kalai, *Polytope skeletons and paths*, in "Handbook of Discrete and Computational Geometry" (J. E. Goodman and J. O'Rourke, eds.), Chapman & Hall/CRC, Boca Raton, FL, second ed., 2004, pp. 455–476.

[Kal12] ______, *Open problems for convex polytopes I'd love to see solved*, July 2012, Talk on Workshop for convex polytopes, Kyoto, slides available on gilkalai.files.wordpress.com/2012/08/kyoto-3.pdf.

[Kö80] M. Kömhoff, *On a combinatorial problem concerning subpolytopes of stack polytopes*, Geometriae Dedicata **9** (1980), 73–76 (English).

[RG96] J. Richter-Gebert, *Realization Spaces of Polytopes*, Lecture Notes in Mathematics, vol. 1643, Springer, Berlin, 1996.

[She74] G. C. Shephard, *Subpolytopes of stack polytopes*, Israel Journal of Mathematics **19** (1974), 292–296 (English).

[Zie08] G. M. Ziegler, *Nonrational configurations, polytopes, and surfaces*, Math. Intelligencer **30** (2008), 36–42.